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On the Spherical Norm of Doob and Seidel's Class

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Let $D(\rho)$ be the Doob's class containing all functions $f(z)$ analytic in the unit disk Δ such that $f(0)=0$ and $\liminf|f(z)|\geqslant 1$ on an arc A of $\partial\Delta$ with length $|A|\geqslant\rho$. It is first proved that if $f\in D(\rho)$ then the spherical norm $\|f\| = \sup_{z\in\Delta}(1-|z|^2)|f'(z)|/(1+|f(z)|^2) \geqslant C_1 \sin(\pi-(\rho/2))/(\pi-(\rho/2))$, where $C_1=\lim_{n\to\infty}\|z^n\|$ and $1/2<C_1<2/e$. Next, U represents the Seidel's class containing all non-constant functions $f(z)$ bounded analytic in Δ such that $|f(e^{i\theta})|=1$ almost everywhere. It is proved that $\inf_{f\in U}\|f\|=0$, and if f has either no singularities or only isolated singularities on $\partial\Delta$, then $\|f\|\geqslant C_1$. Finally, it is proved that if f is a function normal in Δ, namely, the norm $\|f\|<\infty$, then we have the sharp estimate $\|f^p\|\leqslant p\|f\|$, for any positive integer p. © 1985 Academic Press, Inc.

1. Introduction

Let $\Delta=\{z: |z|<1\}$ be the unit disk and let $\partial\Delta$ be the boundary of Δ. Following Doob [3, p. 119], we say that a function $f(z)$ analytic in Δ has the property $D(\rho)$ if $f(0)=0$ and for some arc $A\subset\partial\Delta$ of length $|A|\geqslant\rho>0$ we have

$$\liminf|f(P_n)|\geqslant 1, \tag{1}$$

where $\{P_n\}$ is an arbitrary sequence of points in Δ tending to a point on A. For simplicity, we shall say that $\liminf|f(z)|\geqslant 1$ on A to denote the inequality (1) to be true on A.

In [3, p. 120], Doob asked the question whether the set of Bloch norms

$$\|f\|_B=\sup_{z\in\Delta}|f'(z)|(1-|z|^2)$$

has a positive lower bound depending only on ρ for all functions f in the class $D(\rho)$. In a series of our works [5–11], we have answered this question in the affirmative sense. The method we used is based upon a sharpened form of two constant theorem due to Lehto and Virtanen [14]. Instead of Bloch norm, in [14] the authors consider the following spherical norm

$$\|f\|=\sup_{z\in\Delta}\delta(f(z))(1-|z|^2),$$

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where $\delta(f(z)) = |f'(z)|/(1+|f(z)|^2)$ is the spherical derivative of f. Clearly, we have $\|f\| \leqslant \|f\|_B$ for any function.

Note that the notion of the spherical norm $\|f\|$ is different from the definition of norm in the usual sense. For instance, we do not have the norm property that $\|cf\| = c\|f\|$ $(c > 0)$ (see the function $f(z) = z + 2$), where we have $\|f\| = \frac{1}{2}$, but $\|cf\| = c/(1+c)^2 \neq c\|f\|$ $(c \neq 1)$. Of course, the Bloch norm does have this property.

Also note that the numbers of Bloch and spherical norms of a function can be very different; for instance, if $f(z) = 1/(1-z)$, then $\|f\| < \infty$, but $\|f\|_B = \infty$. Thus a property which holds for spherical norm may not be true for Bloch norm; for instance, the spherical norm of power satisfies $\|f^p\| \leqslant p\|f\|$, but this inequality is false for Bloch norm (see Theorem 11).

In view of the problem of Doob, we may therefore ask whether the set of spherical norms $\{\|f\| : f \in D(\rho)\}$ has a positive lower bound depending only on ρ. The answer turns out to be yes. In contrast to [5, Theorem 1], we have the following lower bound of spherical norms in place of Bloch norms for functions in Doob's class.

THEOREM 1. *If $f \in D(\rho)$, then*

$$\|f\| \geqslant C_1 \sin(\pi - (\rho/2))/\pi - (\rho/2)) \tag{2}$$

which is sharp when ρ tends to 2π, where C_1 is the unique solution of the equation

$$F(C) = [1 + (1 + C^2)^{1/2}]/Ce^{(1+C^2)^{1/2}} = 1. \tag{3}$$

Furthermore, this number C_1 can be evaluated by means of the approximation

$$C_1 = \lim_{n \to \infty} |z^n|, \tag{4}$$

and it satisfies

$$\tfrac{1}{2} < C_1 < 2/e. \tag{5}$$

Note that the estimate in (2) is not sharp when ρ is small. A better estimate can be obtained by the same argument as that of [6, Theorem 1]. However, this improvement is not best possible for small ρ. The best one was recently found in [9, Theorem 1] for Bloch norm. Based upon this method, we shall prove the following estimate for spherical norm.

THEOREM 2. *For each $0<\varepsilon<1$, there is a positive number ρ_0 such that each function $f \in D(\rho)$ with $\rho \leqslant \rho_0$, the spherical norm satisfies*

$$\|f\| > 2(1-\varepsilon)(1+\lambda^2)^{-1/2}\left\{\log\frac{1+\cos(\rho/2)}{1-\cos(\rho/2)}\right\}^{-1}, \tag{6}$$

where

$$\lambda = \inf_{e^{i\theta}\in A}\ \sup_{0\leqslant r<1} |f(re^{i\theta})|,$$

which does not hold if the number $(1-\varepsilon)$ is replaced by $\sqrt{2}$.

Note that the estimate in (6) *is much better than that of* (2) *for all sufficiently small ρ. In fact, as $\rho\to 0$,* (6) *becomes*

$$\|f\| > \tfrac{1}{2}(1-\varepsilon)(1+\lambda^2)^{-1/2}\{\log(1/\rho)\}^{-1} = N(\rho)\,\rho, \quad N(\rho)\to\infty,$$

where λ is bounded. Of course, if the number ρ is large then (2) *is better than* (6). *Therefore the above two theorems behave in two extreme cases. Neither of them can be derived from the other.*

2. NORMAL FUNCTIONS

Let G be a domain and let $d\sigma(z)$ be the hyperbolic element of length in G. Following Lehto and Virtanen [14, Theorem 3], we say that a function f meromorphic in G is normal if and only if the following spherical norm is finite:

$$\sup_{z\in G} \delta(f(z))|dz|/d\sigma(z) = C < \infty.$$

With this definition, we shall state the following extension of two constant theorems [14, Theorems 6 and 7].

THEOREM A. *Let $f(z)$ be a function normal in the upper half-plane H with norm $C<\infty$. Let S be a segment on the real axis and let T_α be the circle passing through the endpoints of S and containing the interior angle α. If m and M are two positive constants such that $m<M$ and*

$$\limsup |f(z)| \leqslant m \quad \textit{on } S \quad \textit{and} \quad \max|f(z)| = M \quad \textit{on } T_\alpha\cap H,$$

then we have

$$m \geqslant M\exp\left\{-\frac{C(\pi-\alpha)}{2\sin\alpha}\left(M+\frac{1}{M}\right)\right\}. \tag{7}$$

We shall now transform the above theorem from the upper half-plane to the unit disk. For this, we first note that the hyperbolic element of length in the disk becomes (see [14, p. 55])

$$d\sigma(z) = |dz|/(1-|z|^2),$$

so by the property of conformal invariance (see [14, p. 55(5)]) we find that the spherical norms in the half-plane and the unit disk are the same, that is, $C = \|f\|$.

To apply Theorem A, we let G_α be the circular subdomain of H bounded by S and T_α. Then G_α can also be defined in term of harmonic measure, namely,

$$G_\alpha = \{w \colon \omega(w, S, H) > \alpha/\pi\},$$

where $\omega(w, S, H)$ is harmonic in H and assumes the values 1 and 0 on S and $\partial H - S$, respectively (see Hille [4, p. 408]). Since harmonic measures are conformally invariant, it follows that the image of G_α in Δ is the same as the following lens-shaped domain (see [5, p. 232])

$$L(\alpha, A) = \{z \colon \omega(z, A, \Delta) > \alpha/\pi\}, \tag{8}$$

where $z = z(w)$ is a conformal mapping from H onto Δ and $A = z(S)$ is an arc on $\partial\Delta$. Let the length of the arc A be ρ, namely, $A = \{e^{i\theta}\colon \theta_0 \leqslant \theta \leqslant \theta_0 + \rho\}$. Then by Poisson's formula (see Ahlfors [1, p. 166]), we can represent the harmonic measure by

$$\omega(re^{i\theta}, A, \Delta) = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+\rho} \frac{(1-r^2)\,dt}{1+r^2-2r\cos(\theta-t)}.$$

In particular, we have $\omega(0, A, \Delta) = \rho/2\pi$. This shows that the boundary $\partial L(\alpha, A)$ contains the origin provided $\alpha = \rho/2$. Substituting $\alpha = \rho/2$ and $C = \|f\|$ into (7), we obtain the following analogue of Theorem A.

THEOREM 3. *Let A be an arc on $\partial\Delta$ with length $|A| = \rho$ and let $L = L(\rho/2, A)$ be the lens-shaped domain defined in* (8). *Then the boundary ∂L contains the origin. Furthermore, if g is a function normal in Δ such that*

$$\limsup |g(z)| \leqslant m < M \text{ on } A \quad \text{and} \quad \max|g(z)| = M \text{ on } \partial L - A,$$

then

$$m \geqslant M \exp\left\{-\frac{\|g\|(\pi-(\rho/2))}{2\sin(\rho/2)}\left(M+\frac{1}{M}\right)\right\}. \tag{9}$$

3. Proof of Theorem 1

With the help of Theorem 3, we are now able to prove the main part of Theorem 1 and we postpone the numerical computation of the transcendental number C_1 to Section 7.

Let $f \in D(\rho)$ and let $g = 1/f$. Then clearly the spherical derivatives of f and g are the same so that the norms $\|f\| = \|g\|$.

To prove the assertion (2), we suppose on the contrary that

$$\|f\| < C_1 \sin(\pi - (\rho/2))/(\pi - (\rho/2)). \tag{10}$$

It then follows that both f and g are normal in Δ. Furthermore, from (1), we have

$$\limsup |g(z)| \leqslant 1 \qquad \text{on } A, \qquad \text{where } |A| = \rho.$$

Let $L = L(\rho/2, A)$ and $\max|g(z)| = M$ on L. Then by substituting $m = 1$ into (9) and combining with (10), we obtain

$$(M \log M)/(M^2 + 1) < C_1/2. \tag{11}$$

Denote by $h(M)$ the function on the left-hand side of (11). Letting $h'(M) = 0$, we find that the maximum of $h(M)$ occurs at the point M_1 such that

$$M_1^2 + 1 - (M_1^2 - 1) \log M_1 = 0 \quad \text{and} \quad h(M_1) = M_1/(M_1^2 - 1). \tag{12}$$

Denote by $x = 2h(M_1)$, then $M_1 = [1 + (1 + x^2)^{1/2}]/x$. Substituting M_1 into the first equality of (12), we obtain

$$F(x) = [1 + (1 + x^2)^{1/2}]/xe^{(1+x^2)^{1/2}} = 1,$$

which is the same as Eq. (3). Since

$$F'(x) = -[1 + x^2 + (1 + x^2)^{1/2}]/x^2 e^{(1+x^2)^{1/2}} < 0,$$

it follows that the function $F(x)$ is monotonically decreasing on the positive real axis, so that the equation $F(x) = 1$ has a unique solution. This yields that

$$h(M_1) = C_1/2 \qquad \text{and} \qquad M_1 = [1 + (1 + C_1^2)^{1/2}]/C_1 > 1.$$

We now consider an arbitrary subarc $B \subset A$ of length β. We then denote by $M(\beta)$ the maximum of g on the lens $L(\beta/2, B)$. Then the inequality (11) says that the value $M(\beta)$ lies in one of the intervals $[1, M_1)$ and (M_1, ∞). Since the function g is continuous, the set of values of $M(\beta)$ must be connected. Letting $\beta \to 0$, we see that $M(\beta) \to 1$. Hence we must have $M(\beta) < M_1$ for each β. However, the one $M(\rho) = \infty$ because $g(0) = \infty$ and the boundary $\partial L(\rho/2, A)$ contains the origin. This contradiction proves the assertion (2).

4. More on the Two Constant Theorem

To prove Theorem 2, we shall need to study some more extensions of the two constant theorems described in Theorem A. For this, we now introduce an alternate definition of lens-shaped domain. In view of the notions in Theorem A, we let $\beta < \pi$ be the angle between the segment S and the tangent of the circle T_α at one of the endpoints of S. Since T_α contains the interior angle α, it follows from a basic geometric property that either $\beta = \alpha$ or $\pi - \alpha$ depending on the center of T_α lying inside or outside of H, respectively. The first case has been considered in Theorems 1 and 3. To prove Theorem 2, we shall need the second case. In this case, our definition of lens-shaped domain is different from the one in (8), namely,

$$L^*(\beta, A) = \{z \colon \omega(z, A, A) > (\pi - \beta)/\pi\}. \tag{13}$$

With this definition and the help of Theorem 3, we can now easily prove the following version of two constant theorem due to Pommerenke [16, Theorem 9.1].

Theorem 4. *Under the hypothesis of Theorem 3, if the lens there is replaced by $L^*(\beta, A)$ defined in (13), and if*

$$M < M^*(\beta) = \left[1 + \left(1 + \left(\frac{\beta\|g\|}{\sin\beta}\right)^2\right)^{1/2}\right] \Big/ \left(\frac{\beta\|g\|}{\sin\beta}\right), \tag{14}$$

then we have

$$|g(z)| \leqslant M_1, \qquad \text{for all} \quad z \in L^*(\beta, A),$$

where $M_1 > m$ is the smallest solution of the equation

$$m = M \exp\left\{-\frac{\beta\|g\|}{2\sin\beta}\left(M + \frac{1}{M}\right)\right\}. \tag{15}$$

Proof. Let $h(M)$ be the function on the right-hand side of (15). Then the equation $h(M) = m$ has two roots M_1 and M_2 with $m < M_1 < M_2$.

Letting $h'(M)=0$, we find that the function $h(M)$ attains its maximum at the point $M^*(\beta)$ defined in (14), so that

$$M_1 < M^*(\beta) < M_2. \tag{16}$$

In view of the inequality (9) with β in place of $\pi-(\rho/2)$, we see that the range of M must be either $M \leqslant M_1$ or $M \geqslant M_2$. It then follows from (14) and (16) that

$$M \leqslant M_1, \qquad \text{where } M=\max|g(z)|, \qquad \text{for } z \in L^*(\beta, A).$$

This yields the assertion.

We shall now improve Theorem 4 by omitting the hypothesis that $M < M^*(\beta)$. This result will be needed, so we formulate as follows.

THEOREM 5. *The assertion of Theorem* 4 *is true without the assumption that* $M < M^*(\beta)$.

Proof. As in Theorem 3, we write

$$\max|f(z)|=M \qquad \text{on } \partial L^* - A, \qquad \text{where } L^*=L^*(\beta, A).$$

Since $\limsup|f(z)| \leqslant m < M$ on A, it follows from the maximum principle that

$$\max|f(z)|=M \qquad \text{on } L^*(\beta, A).$$

If $M \leqslant M_1$, then we have

$$|f(z)| \leqslant M \leqslant M_1, \qquad \text{for } z \in L^*(\beta, A).$$

This gives the assertion.

On the other hand, if $M > M_1$, then there can be chosen a lens domain $L^*(\beta', A) \subset L^*(\beta, A)$, where $\beta' < \beta$, such that

$$\max|f(z)|=M_1 \qquad \text{on } L^*(\beta', A).$$

Substituting M_1 and β' into M and $\pi-(\rho/2)$, respectively, in (9), we obtain

$$m \geqslant M_1 \exp\left\{-\frac{\beta'\|f\|}{2\sin\beta'}\left(M_1+\frac{1}{M_1}\right)\right\}. \tag{17}$$

Since $\beta' < \beta$ and the function $\theta/\sin\theta$ is strictly increasing on the interval $[0, \pi]$, it follows from (17) first and then (15) that

$$m > M_1 \exp\left\{-\frac{\beta\|f\|}{2\sin\beta}\left(M_1+\frac{1}{M_1}\right)\right\}=m,$$

which is absurd. Thus the case $M > M_1$ is impossible provided M_1 is the smallest solution of (15). This proves the theorem.

With the help of Theorem 5, we are now able to prove the following result, which will be needed in the proof of Theorem 2.

THEOREM 6. *Let A be an arc on $\partial\Delta$ with length $|A| \geqslant \rho$, and let $L^*(\beta, A)$ be the lens domain defined in* (13). *If f is a function normal in Δ and has the $D(\rho)$ property on A, then*

$$|f(z)| \geqslant 1/M_1, \qquad \textit{for all} \quad z \in L^*(\beta, A), \tag{18}$$

where $M_1 > 1$ is the smallest solution of Eq. (15) (*with $m = 1$*).

Proof. Let $g(z) = 1/f(z)$, then clearly g and f have the same spherical norm $\|g\| = \|f\|$. In view of (1), the $D(\rho)$ property of f implies that

$$\limsup |g(z)| \leqslant 1 \qquad \text{on } A.$$

Applying Theorem 5 with the substitution $m = 1$ into (15), we obtain

$$|g(z)| \leqslant M_1 \quad \text{or} \quad |f(z)| \geqslant 1/M_1, \quad \text{for all } z \in L^*(\beta, A),$$

where M_1 is the smallest solution of Eq. (15). This proves the assertion.

5. PROOF OF THEOREM 2

Let $f \in D(\rho)$ and let $0 < \varepsilon < 1$ be given. To prove the assertion (6), we suppose on the contrary that for all sufficiently small $\rho > 0$, the spherical norm

$$\|f\| \leqslant 2(1-\varepsilon)(1+\lambda^2)^{-1/2}\, d(\rho), \; d(\rho) = \left\{\log \frac{1+\cos(\rho/2)}{1-\cos(\rho/2)}\right\}^{-1}. \tag{19}$$

It follows from (19) that the function f is normal in Δ. Since the function f has the $D(\rho)$ property on A, by applying Theorem 6 we obtain the inequality (18), where M_1 is the smallest solution of Eq. (15). This number depends only on β and $\|f\|$. From (19), we see that the norm $\|f\|$ depends on the number ρ, and therefore we may write $M_1 = M_1(\beta, \rho)$. It then follows from (15) and (19) that

$$\lim_{\rho \to 0} M_1(\beta, \rho) = 1, \qquad \text{where } 0 < \beta < \pi \text{ is fixed.} \tag{20}$$

Combining (18) with (20), we can find a positive number ρ_0 depending on ε such that for each $\rho \leqslant \rho_0$, we have

$$|f(z)| > 1 - \varepsilon, \quad \text{for all } z \in L^*(\beta, A), \quad \text{where } |A| = \rho. \tag{21}$$

In view of the number λ defined in (6), by applying a rotation, we may, without loss of generality, assume that

$$\sup_{0 \leqslant x < 1} |f(x)| = \lambda. \tag{22}$$

Now, let $S(a, b)$ and $X(a, b)$ be the spherical and chordal distance between a and b (see [1, p. 218]). Then by the hypothesis $f(0) = 0$ and the definition of spherical norm, we obtain

$$\begin{aligned} X(f(0), f(x)) &= 2|f(x)|/(1 + |f(x)|^2)^{1/2} \leqslant S(f(0), f(x)) \\ &= 2 \int_0^x \delta(f(t))\, dt \leqslant 2\|f\| \int_0^x \frac{dt}{1 - t^2} \\ &= \|f\| \log \frac{1 + x}{1 - x}. \end{aligned} \tag{23}$$

Substituting $x = \cos(\rho/2)$ into (23) and then applying (19) and (22), we get

$$|f(\cos(\rho/2))| \leqslant 1 - \varepsilon. \tag{24}$$

Since (20) holds for each $\beta < \pi$, by choosing $\beta > (\pi + \rho_0)/2$, we see that the point $\cos(\rho/2)$ lies within the lens $L^*(\beta, A)$, where $|A| = \rho \leqslant \rho_0$. In this case, the inequality (24) contradicts (21). This proves the assertion (6).

It remains to show that the constant $(1 - \varepsilon)$ on the right-hand side of (6) cannot be replaced by $\sqrt{2}$. To do this, we need only consider the following function

$$\phi(z) = (d(\rho)^{-2} + \pi^2)^{-1/2} \log \frac{1 + z}{1 - z},$$

where the number $d(\rho)$ is defined in (19). By a simple computation, we see that

$$|\phi(e^{i\alpha})| = 1, \qquad \text{for } \alpha = \pm\rho/2,$$

and therefore from the symmetric consideration we conclude that the function ϕ has the $D(\rho)$ property on the arc $A = \{e^{i\theta}\colon -\rho/2 \leqslant \theta \leqslant \rho/2\}$. Furthermore, the mapping $(1 + z)/(1 - z)$ carries the disk Δ onto the right half-plane H and carries the diameter passing through the point $e^{i\rho/2}$ onto the half circle in H passing through these three points $\cot \rho/4$, 1, and $\tan \rho/4$. It follows that

$$\max_{0<r<1} |\phi(re^{i\rho/2})| = |\phi(e^{i\rho/2})| = 1.$$

This shows that the number $\lambda = 1$ in (6).

Finally, it is easy to see that the norm $\|f\|$ occurs at the origin, so that

$$\|\phi\| = 2(d(\rho)^{-2} + \pi^2)^{-1/2} = 2d(\rho) - o(\rho),$$

where $o(\rho) > 0$ and $o(\rho) \to 0$ as $\rho \to 0$. This shows that the number $(1-\varepsilon)$ in (6) cannot be replaced by $\sqrt{2}$ and the proof is complete.

Note that the estimate in (6) can be improved a little depending on the location of the radius in which the function f is bounded by λ. To see this, let this radius separate the arc A into two subarcs of lengths ρ_1 and ρ_2, where $\rho_1 + \rho_2 = |A| \geqslant \rho$ and $\rho_1 \leqslant \rho_2$. Then by the same argument, we obtain

$$\|f\| > 2(1-\varepsilon)(1+\lambda^2)^{-1/2}\left\{\log \frac{1+\cos\rho_2}{1-\cos\rho_2}\right\}^{-1},$$

which is better than that of (6) because $\rho_2 \geqslant \rho/2$. From this, we see that the best case occurs when $\rho_1 \to 0$ and $\rho_2 \to \rho$, in other words, if f is bounded by λ on one of the radii passing through the endpoints of A.

As a consequence of Theorems 1 and 2, we obtain immediately the following combining result.

COROLLARY 1. *If $f \in D(\rho)$ and if $0 < \varepsilon < 1$ is given, then*

$$\begin{aligned} \|f\| &\geqslant K_1(\rho), && \textit{for all } 0 < \rho < 2\pi, \\ &\geqslant \max(K_1(\rho), K_2(\rho, \varepsilon, \lambda)), && \textit{for } \rho \leqslant \rho_0, \end{aligned}$$

where ρ_0 depends on ε, and $K_1(\rho)$ and $K_2(\rho, \varepsilon, \lambda)$ denote the numbers on the right-hand side of (2) *and* (6), *respectively.*

To end this section, we shall pose a problem about the possible improvement of Theorem 2. In view of the sharp estimate of Bloch norm in [9, Theorem 1], we have under the hypothesis of Theorem 2 that

$$\|f\|_B > 2(1-\varepsilon)\, d(\rho),$$

which does not hold when $\varepsilon = 0$, due to the function ϕ defined before. Comparing this estimate with (6), we see that the main difference between the lower bounds of Bloch and spherical norms occurs in the factor

$(1+\lambda^2)^{-1/2}$. This factor creates the difficulty for finding the best lower bound of the spherical norm. Whether the number $(1-\varepsilon)$ in (6) can be improved by the number $\sqrt{2}\,(1-\varepsilon)$, in general, we do not know. Of course, if this improvement is true then it is certainly best possible.

6. Seidel's Class U

Following Collingwood and Lohwater [2, p. 107], we shall denote by U the class of all non-constant bounded analytic functions $f(z)$ for which the radial limits $f(e^{i\theta})$ exist and have modulus 1 for almost all points $e^{i\theta}$. Functions of this kind can be represented by (see Seidel [17, Theorem 1])

$$f(z) = e^{i\theta} B(z)\, S(z), \tag{25}$$

where

$$B(z) = z^k \prod_{i=1}^{\infty} \frac{|a_i|}{a_i} \frac{a_i - z}{1-\bar{a}_i z}, \quad a_i \neq 0 \text{ and } a_i \in \Delta,$$

is the Blaschke product extended over all zeros of f, and

$$S(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)\right)$$

is the singular inner function for which the distribution $\mu(\theta)$ is decreasing and $\mu'(\theta) = 0$, almost everywhere on $(-\pi, \pi)$.

In this section, we shall study the spherical norm for functions in the class U. We begin with proving the following upper and lower bounds of norms in the class U.

Theorem 7. $\sup_{f\in U}\|f\| = 1$ *and* $\inf_{f\in U}\|f\| = 0$.

Proof. Since f is bounded by 1 in Δ, it follows from the Schwarz's lemma (see Ahlfors [1, p. 136]) that

$$(1-|z|^2)|f'(z)| \leqslant 1 - |f(z)|^2 \leqslant 1,$$

so that the spherical norm $\|f\| \leqslant \|f\|_B \leqslant 1$. This together with the fact that $\|I\| = 1$, where $I(z) = z \in U$, yields the first equality.

To prove the second equality, we let $\varepsilon > 0$ be given, and let $\{p_n\}$ be a sequence of points in Δ such that the complement $G = \Delta - U\{p_n\}$ contains no disks of radius ε. Denote by f a conformal mapping from Δ onto the universal covering surface of G. Then clearly the function $f \in U$. Applying

Bloch's theorem (see Hille [4, Theorem 17.7.1]), we find that the Bloch norm

$$\|f\|_B < b\varepsilon, \qquad \text{where } b > 0 \text{ is independent from } f.$$

Choosing a sequence of $\varepsilon_n \to 0$, we obtain a sequence of functions $f_n \in U$ such that $\|f_n\| \to 0$. This yields the second equality, and the proof is complete.

According to the above theorem, we know that, in general, the set of norms of functions in the class U cannot be bounded away from zero. This gives the motivation to define a subclass of U for which the set of norms has a positive lower bound. For this, we say that a function $f \in U_1$ if $f \in U$ and if f has at most one singular point on $\partial\Delta$.

THEOREM 8. *If $f \in U_1$, then the norm $\|f\| \geq C_1$, where C_1 is defined in Theorem* 1.

Proof. According to the hypothesis and the representation in (25), we see that the radial limits $f(e^{i\theta})$ exist and have modulus 1 for all points $e^{i\theta}$ on $\partial\Delta$ with at most one exception. It follows that the function $f \in D(\rho)$ for each $\rho < 2\pi$. Substituting the limit $\rho \to 2\pi$ into (2), we obtain the assertion $\|f\| \geq C_1$.

With the help of Theorem 8, we shall now prove the following criterion of the class U_1 in term of the function $F(C)$ defined in Eq. (3).

THEOREM 9. *If $f \in U_1$, then*

$$1 \geq F(C), \qquad \textit{where } C = \|f\|. \tag{26}$$

Proof. In view of Theorem 1, we know that the function F is monotonically decreasing on $(0, \infty)$ and the number C_1 is the unique solution of the equation $F(x) = 1$. Thus, if (26) were false, we would have

$$F(C) > 1 = F(C_1), \qquad \text{so that } \|f\| = C < C_1,$$

which contradicts Theorem 8. This proves the theorem.

Note that as far as the application of Theorem 1 is concerned, the assertion of Theorem 8 cannot be improved by allowing the function f there to have more than one singular point on $\partial\Delta$. This leads us to ask the question whether such an improvement is possible. The answer turns out to be yes, as will be seen from the following result which makes no restriction on the number of singularities.

THEOREM 10. *If $f \in U$ and if f has an isolated singularity P on $\partial\Delta$ such* *that f has no zeros in a neighborhood of P, then the norm $\|f\| \geqslant C_1$, where C_1 is defined in Theorem* 1.

To prove the above theorem, we shall first prove the following particular case.

LEMMA 1. *Let $f_1(z) = \exp[-(1+z)/(1-z)]$, then*

$$\|f_1^n\| = C_1, \qquad \textit{for any } n > 0.$$

Proof. We first write

$$z = x + iy \quad \text{and} \quad t = [1 - (x^2 + y^2)]/[(1-x)^2 + y^2],$$

and denote by $H(t)$ the following horocycle tangent to $\partial \Delta$ at $z = 1$,

$$[x - (t/(1+t))]^2 + y^2 = 1/(1+t)^2.$$

By a simple computation, we have for $z \in H(t)$

$$|f_1(z)| = e^{-t}$$

and

$$(1 - |z|^2)\, \delta(f(z)) = 2te^{-t}/(1 + e^{-2t}) = h(t).$$

Let C be the maximum of $h(t)$ over $[0, \infty)$; then by setting $h'(t) = 0$, we get

$$1 - t + (1+t)\, e^{-2t} = 0, \qquad \text{where } t > 1, \tag{27}$$

so that

$$C = \max h(t) = (t_1^2 - 1)^{1/2} \qquad \text{or} \qquad t_1 = (1 + C^2)^{1/2}.$$

Substituting t_1 into (27), we find that C satisfies the equation $F(C) = 1$, defined in Theorem 1. Since this equation has a unique solution, we must have $C = C_1$. This proves the lemma for $n = 1$. The general case can be proved by replacing t by nt.

Proof of Theorem 10. According to the hypothesis, we may assume that the function f has an isolated singularity at the point $z = 1$. Since $f \in U$ and f has no zeros in a neighborhood of the singularity $z = 1$, it follows from (25) that the function f can be represented by

$$f(z) = f_1^n(z)\, f_2(z), \qquad \text{for some } n > 0, \tag{28}$$

where f_1 is defined in Lemma 1 and f_2 is analytic in a neighborhood of $z=1$. Moreover, since we may assume $n=1$ as in Lemma 1, we then have

$$\begin{aligned}\delta(f_1(z)) &= |f_2(z)\,f'(z) - f(z)\,f_2'(z)|/(|f_2(z)|^2 + |f(z)|^2),\\ &\leqslant \delta_1(z) + \delta_2(z),\end{aligned} \tag{29}$$

where $\delta_1(z) = |f_2(z)\,f'(z)|/d(z)$, $\delta_2(z) = |f(z)\,f_2'(z)|/d(z)$, and $d(z) = |f_2(z)|^2 + |f(z)|^2$. By the condition of f_2 described in (28), we get

$$\lim_{z\to 1} |f_2(z)| = 1 \qquad \text{and} \qquad \lim_{z\to 1} f_2'(z) = f_2'(1).$$

It follows that for z tending to 1 along the maximum horocycle $H(t_1)$, where $t_1 = (1 + C_1^2)^{1/2}$, we have

$$\lim_{z\to 1} (1 - |z|^2)\,\delta_1(z) = \lim_{z\to 1} (1 - |z|^2)\,\delta(f(z)) \leqslant \|f\|,$$

and

$$\lim_{z\to 1} (1 - |z|^2)\,\delta_2(z) = 0. \tag{30}$$

Owing to (29), (30), and Lemma 1, we obtain the following desired result:

$$C_1 = \lim_{z\to 1} (1 - |z|^2)\,\delta(f_1(z)) \leqslant \|f\|, \qquad \text{where } z \in H(t).$$

As a consequence o Theorem 10 we obtain the following analogue of Theorem 9.

COROLLARY 2. *Under the hypothesis of Theorem* 10, *we have the inequality* (26).

The proof is the same as in Theorem 9 and we omit the details.

7. COMPLETE PROOF OF THEOREM 1

With the help of Lemma 1 and Theorem 8, we can now easily finish the proof of Theorem 1. We first verify the assertion that the inequality (2) is sharp when ρ tends to 2π. In other words, inequality (2) cannot be improved by $\|f\| \geqslant C$ for any $C > C_1$ and any function $f \in D(2\pi)$. This is true because the function f_1 defined in Lemma 1 belongs to the class $D(2\pi)$ and its norm reaches the minimum $\|f_1\| = C_1$.

It remains to prove that C_1 satisfies (4) and (5). For this, we let $z = re^{i\theta}$ and let $I(z) = z$ be the identity function. Then clearly we have

$$(1-|z|^2)\,\delta(I^n(z)) = nr^{n-1}(1-r^2)/(1+r^{2n}) = g(r).$$

Let the maximum of $g(r)$ occur at $r^* = (1-2t/n)^{1/2}$; then the norm

$$\|I^n\| = g(r^*) = 2t(1-2t/n)^{(n-1)/2}/(1+(1-2t/n)^n) = h_n(t),$$

where t depends on n. It follows that

$$h(t) = \lim_{n\to\infty} h_n(t) = 2te^{-t}/(1+e^{-2t}).$$

This function is the same as that of Lemma 1 and therefore from (27) we obtain

$$\lim_{n\to\infty} \|I^n\| = h(t) \leqslant C_1, \qquad \text{where } C_1 = \max_{t\geqslant 0} h(t). \tag{31}$$

On the other hand, by virtue of Theorem 8, we have $\|I^n\| \geqslant C_1$, for each $n = 1, 2, \ldots$, so that

$$\lim_{n\to\infty} \|I^n\| \geqslant C_1.$$

This together with (31) yields the assertion (4).

To prove (5), we first note that the number $C_1 = \max h(t)$, so that $C_1 > h(1) > \frac{1}{2}$. On the other hand, by the same argument as before, we have

$$\lim_{n\to\infty} \|I^n\|_B = 2/e, \qquad \text{so that } C_1 < 2/e.$$

Hence we obtain $\frac{1}{2} < C_1 < 2/e$. This proves (5) and the proof of Theorem 1 is complete.

8. The Power of Normal Functions

In this last section, we shall study a basic property about the norm of power of normal functions. Recall a definition from Lappan [13, Theorem 10]. We say that a function f analytic in Δ is uniformly normal if its Bloch norm is finite. Functions of this kind have been called Bloch functions by Pommerenke [15]. The properties of normal and uniformly normal functions behave differently in many aspects. For instance, the sum of uniformly normal functions is again uniformly normal, but this property is not true for normal functions according to Lappan [12]. In other words, the class of uniformly normal functions is closed under the sum operation, but not for the class of normal functions. In [12, Theorem 1], Lappan proved that the class of normal functions is not closed under the product operation. This leads us to question whether it is closed under the power operation. The answer turns out to be yes for normal functions, but no for uniformly normal functions as will be seen from the following result.

THEOREM 11. *If f is a function normal in Δ, then we have*

$$\|f^p\| \leqslant p\|f\|, \qquad \textit{for each positive integer } p, \tag{32}$$

which is sharp, so f^p is normal. But there is a uniformly normal function ϕ such that $\|\phi^p\|_B = \infty$, for each $p \geqslant 2$.

Proof. To prove (32), we let the function $h = f^p$. We shall first consider that the norm $\|h\|$ is attained at an interior point $z_0 \in \Delta$, namely,

$$\|h\| = \frac{|h'(z_0)|(1-|z_0|^2)}{1+|h(z_0)|^2}.$$

There are two cases to be considered: either $|f(z_0)| \geqslant 1$ or not. By a simple computation, the first case yields

$$\frac{|f(z_0)|^{p-1}}{1+|f(z_0)|^{2p}} \leqslant \frac{1}{1+|f(z_0)|^2}. \tag{33}$$

Multiplying by $p|f'(z_0)|(1-|z_0|^2)$ on both sides of (33), we obtain

$$\|f^p\| = \|h\| \leqslant \frac{p|f'(z_0)|(1-|z_0|^2)}{1+|f(z_0)|^2} \leqslant p\|f\|,$$

which gives (32) in the first case.

Turning to the second case, we let $H = 1/h$, then we have $\|H\| = \|h\|$. Since the second case requires that $|1/f(z_0)| \geqslant 1$, and the function $H = (1/f)^p$, by considering $1/f$ in place of f in the first case, we obtain

$$\|f^p\| = \|h\| = \|H\| \leqslant p\|1/f\| = p\|f\|.$$

This proves (32) in the second case, so that (32) holds in both cases.

It remains to consider the case that the norm $\|h\|$ is not attained at any interior point. This, however, is easy because in this case there is a sequence of points $z_n \in \Delta$ such that

$$h^*(z_n) = \frac{|h'(z_n)|(1-|z_n|^2)}{1+|h(z_n)|^2} \to \|h\|, \qquad \text{as } n \to \infty.$$

By the same argument, we find that

$$h^*(z_n) \leqslant p\|f\|, \qquad n = 1, 2, \ldots,$$

and therefore $\|h\| \leqslant p\|f\|$. This proves (32).

We shall now prove that the estimate in (32) is sharp. For this, we consider the functions

$$f = \phi^2 \quad \text{and} \quad h = \phi^{2p}, \quad \text{where } \phi(z) = \log \frac{1+z}{1-z}.$$

Clearly, we have

$$h^*(z) = \frac{|h'(z)|(1-|z|^2)}{1+|h(z)|^2} = \left\{\frac{2p(1-|z|^2)}{|1-z^2|}\right\}\left\{\frac{2|\phi(z)|^{2p-1}}{1+|\phi(z)|^{4p}}\right\}. \tag{34}$$

We shall prove that the norm of h satisfies

$$\|h\| = 2p. \tag{35}$$

Let the maximum of (34) occur at a point z_0. Since the function $\phi(z)$ is symmetric with respect to the real axis, it follows that the point z_0 must be real, so that the first factor on the right-hand side of (34) becomes $2p$. Turn to the second factor, again, by considering the reciprocal $H = 1/h$, we may, without loss of generality, assume that

$$|h(z_0)| \geqslant 1, \qquad \text{so that } |\phi(z_0)| \geqslant 1.$$

This in turn implies

$$2|\phi(z_0)|^{2p-1} \leqslant 1 + |\phi(z_0)|^{4p}.$$

Combining with the first fact $2p$, we obtain $\|h\| \leqslant 2p$. To prove the converse, we observe that the set $S = \{z\colon z \text{ is real and } |\phi(z)| = 1\}$ contains two points. It then follows from (34) that

$$\|h\| \geqslant h^*(z) = 2p, \qquad \text{for } z \in S.$$

This together with $\|h\| \leqslant 2p$ yields (35).

In view of the definition of f and h, and the equality (35), we obtain

$$\|f\| = 2 \qquad \text{and} \qquad \|h\| = 2p = p\|f\|.$$

This shows the sharpness of (32).

Finally, we shall prove that the function ϕ is uniformly normal in Δ, but the power ϕ^p is not. This is easy and in fact we have the Bloch norms

$$\|\phi\|_B = 2 \quad \text{and} \quad \|\phi^p\|_B = \infty, \quad \text{for each } p \geq 2.$$

This completes the proof.

We emphasize that the function ϕ does have the property that the spherical norms $\|\phi\| = \|\phi^2\| = 2$ and $\|\phi^{2p}\| = 2p$, $p \geq 1$.

Note that the last part of the above theorem shows that the inequality

$$\|f^p\|_B \leq p\|f\|_B, \tag{36}$$

is false for functions uniformly normal in Δ. Since any function in Seidel's class is uniformly normal in Δ, it is naturally to ask whether the inequality (36) is true for those functions. The answer is affirmative, as will be seen from the following result.

THEOREM 12. *If f is analytic and bounded by M in Δ, then*

$$\|f^p\|_B \leq p\, M^{p-1}\|f\|_B.$$

In particular, if $f \in U$, then (34) *holds.*

The proof is straightforward and we omit the details.

In closing this paper, let us pose the following problem about Theorem 12: What is the smallest constant $B(p, M)$ such that the inequality

$$\|f^p\|_B \leq B(p, M)\|f\|_B$$

holds for all functions f analytic and bounded by M in Δ. The same question can also be asked for the spherical norm.

In view of the function $f_1 \in U$ defined in Lemma 1 and the identity function $I(z) = z \in U$, we have that

$$\|f_1^p\|_B = \|f_1\|_B \qquad \text{and} \qquad \|I^p\|_B < \|I\|_B.$$

This leads to question whether it is true that the constant $B(p, 1) = 1$ for all functions in Seidel's class.

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